# Phase transition and hysteresis loop in structured games with global updating

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We present a global payoff-based strategy updating model for studying cooperative behavior of a networked population. We adopt the Prisoner's Dilemma game and the snowdrift game as paradigms for characterizing the interactions among individuals. We investigate the model on regular, small-world, and scale-free networks, and find multistable cooperation states depending on the initial cooperator density. In particular for the snowdrift game on small-world and scale-free networks, there exist a discontinuous phase transition and hysteresis loops of cooperator density. We explain the observed properties by theoretical predictions and simulation results of the average number of neighbors of cooperators and defectors, respectively. Our work indicates that individuals with more neighbors have a trend to preserve their initial strategies, which has strong impacts on the strategy updating of individuals with fewer neighbors; while the fact that individuals with few neighbors have to become cooperators to avoid gaining the lowest payoff plays significant roles in maintaining and spreading of cooperation strategy.

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## I. INTRODUCTION

Understanding the dynamics of complex systems in the perspective of networks has drawn growing interests recently. Systems composed of interacting individuals can be mapped into complex networks with individuals occupying nodes and connections representing interactions among individuals. From this viewpoint, networks are ubiquitous in nature and human society. Many reported observations indicate that real network topologies are neither regular nor purely random, but somewhere in between [1,2]. There is an obvious need to model dynamical processes on complex networks to better mimic and understand plentiful dynamical behaviors of natural and social systems [3,4].

Social dilemma games, as general metaphors for studying cooperative behaviors, have received much attention in the past decades [5,6]. In particular, the combination of traditional iteration games and network theory has provided new access to investigate varieties of social and economical dynamics as well as biological behaviors [7]. Games on networks usually abandon the well-mixed population, so that interactions only exist among neighboring individuals. Two simple games, the Prisoner's Dilemma game (PDG) and snowdrift game (SG), have been studied on regular lattices [8], small-world and scale-free networks [9]. The last two networks possess short average distances with high clustering and high heterogenous degree distributions, respectively. It is found that structures in populations play surprisingly positive roles in the organization and domination of cooperative behaviors, compared to the well-mixed case. Very recently, evolutionary games on finite-size networks have drawn growing interests, since in this case stochastic effects cannot be overlooked [10,11]. In this framework, some underlying mechanisms have been investigated, such as the death-birth updating [12,13], asymmetry between interaction and replacement [14], and coevolution of strategy and structure [15], to better explore the dynamics of network games. Besides, the influences of unequal interactions on the dynamics of games have been considered [16].

In most game models, interactions of playing games and information collection for updating the strategy of each individual are restricted to its neighborhood, and the interaction and strategy updating networks are the same. However, as pointed out in Ref. [14], the two networks may be different in some cases. Furthermore, in social and economical systems, the communication, i.e., exchange of strategy and payoff information can occur not only through playing games, but also through many other ways in the modern society, such as broadcasting, the Internet, and other communication tools. In other words, information for making decisions and updating is often much more global than the interaction restriction of playing games. In this point of view, we propose a structured game model with a global strategy updating process, i.e., the individual with the poorest payoff in the network will switch their strategy. We aim to explore the cooperative behavior influenced by the global updating mechanism, which has not been considered so far. We adopt PDG and SG as typical paradigms on different network structures, including regular, small-world, and scale-free networks. Interestingly, we find multistable states, discontinuous phase transition, and hysteresis loops of cooperator density in the networks. We also provide some analysis and explanations for the obtained results. Hysteresis behaviors have been found in other dynamics, such as the ferromagnetic dynamics, but have not been reported in game dynamics. Our work may shed some new light in understanding the cooperative behaviors in the structured games.

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## II. MODEL

PDG and SG, with two strategies, C and D, played by two players (individuals) can be described by a payoff matrix:

$$\begin{array}{ccc} & \mathbf{C} & \mathbf{D} \\ \mathbf{C} & R & S \\ \mathbf{D} & T & P \end{array},$$

where the entities represent the payoffs for the row players, $R$ is the reward for mutual cooperation, $P$ is the punishment for mutual defection, $T$ is the temptation to defect, and $S$ is the sucker's payoff. PDG and SG mainly differ in the orderings of payoff values, i.e., for PDG, $T>R>P>S$ and for SG, $T>R>S>P$. For simplicity in investigation, an equivalent rescaled payoff matrix for the two games is introduced. For PDG, $R=1$, $S=0$, $T=b$, and $P=0$. For SG, $R=1$, $S=1-r$, $T=1+r$, and $P=0$. Hence each game is controlled by a single parameter, $b$ for PDG and $r$ for SG.

Consider that $N$ individuals are placed on nodes of a certain network. In each round, all pairs of connected individuals play the game simultaneously. The total payoff of each player is the sum over all its encounters. Assume that in each round, one individual with the lowest total payoff will change its strategy, which can also be regarded as the death of the individual together with the birth of a new individual with reverse strategy on the same node. (If the new individual is born with the same strategy, it will as well die next time, since it has the same lowest total payoff as the dead individual.) If there is more than one individual with the same lowest total payoff, then one of them is randomly picked to change its strategy. By repeating the above processes, the system will finally reach a steady cooperator density. We have checked that the steady state can be achieved with only a few flicker individuals, who always have the poorest total payoff, no matter what strategy they adopt. In the following, we focus on the properties of networked games in the steady state.

## III. REGULAR NETWORKS

We first study the above model on some regular structures, including fully connected networks (mean-field), two-dimensional (2D) lattices, and star networks, as shown in Fig. 1. Here, $\rho_c$, as the density of cooperators, turns out to be the most important quantity for characterizing cooperative behaviors. For the fully connected network, we can provide analytical results for $\rho_c$. Assume $N$ individuals with a given game parameter $r$ or $b$. If the system is stable, namely, no individuals change their strategies anymore, the payoff of any individual with C should be equal to that with D (Hereafter, C and D denote C strategy and D strategy, respectively). Hence for SG, we have

$$(N\rho_c - 1) + N(1-\rho_c)(1-r) = N\rho_c(1+r), \quad (1)$$

where the left side is the payoff of C individuals, and the right side is for D individuals. From Eq. (1), $\rho_c = 1-r-1/N \approx 1-r$. For PDG, one can write a similar equation, $N\rho_c - 1 = N\rho_c b$, which gives $\rho_c = 1/[N(1-b)]$. For very large $N$ and $b \neq 1$, $\rho_c \approx 0$. Simulation results are displayed in the left column of Fig. 1, which are exactly the same as the analytical results, independent of the initial cooperator density $\rho_c(0)$. These results are consistent with previously reported results on games in the mean-field case. On 2D lattices, $\rho_c$ shows initial-cooperation-dependent behaviors, as shown in the middle column of Fig. 1. The stable values of $\rho_c$ in the case of $\rho_c(0)=1$ is much higher than that with $\rho_c(0)=0$ for both PDG and SG. Moreover, SG with $\rho_c(0)=0$ shows some step structures for different $r$, while PDG is insensitive to $b$. On star networks, $\rho_c$ is insensitive to the game parameters for both PDG and SG, but for SG it is sensitive to $\rho_c(0)$. On star networks, the hub node plays a dominant role in the organization of strategies. Suppose that in SG, the hub adopts C. In the case of infinite size, if $r \neq 0$, the hub's payoff tends to $\infty$, regardless of others' strategies. Consequently, the leafs with D have higher payoffs than those with C. As a result, $\rho_c$ tends to be 0. On the other hand, if the hub is D, all leafs have to adopt C to avoid being eliminated, leading to $\rho_c = 1$. In PDG, the hub with C is unstable, since after all leafs adopt D, the hub has the lowest payoff. Thus the final steady state exhibits a D hub and leafs with random strategies, reflected by $\rho_c = 0.5$.

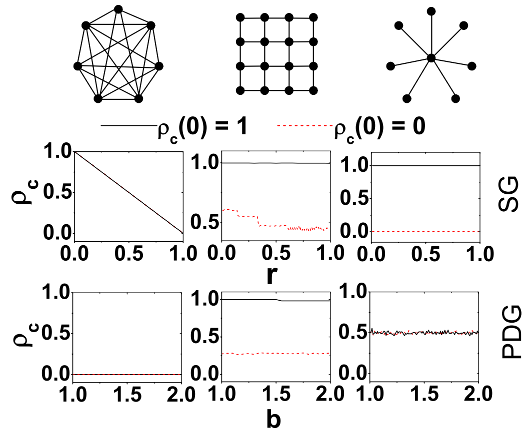


FIG. 1. (Color online). Cooperator density $\rho_c$ for SG and PDG on fully connected networks (left column), 2D lattices (middle column), and star networks (right column). The results of SG and PDG are shown in two rows. $\rho_c(0)$ is the initial cooperator density. The network size is 10 000.

## IV. COMPLEX NETWORKS

In this section, we investigate the model on more realistic networks, including Newman-Watts (NW) small-world [17] and Barabási-Albert (BA) scale-free networks [19]. The NW model is a modified version of the Watts-Strogatz small-world network model [18]. The NW network is constructed by randomly adding edges to a regular ring network. $C_{NW}$ is the coordination number of the regular ring network and $P_{NW}$ is the probability of each node to receive a new edge. In the BA model, there are $m$ nodes initially. At each time step, a new node with $m$ edges is added and preferentially attached to $m$ existing nodes with probability proportional to the degrees of existing nodes. The minimum degree $k_{\min}$ is $m$. In PDG, similar to the case of regular networks, $\rho_c$ is insensi-

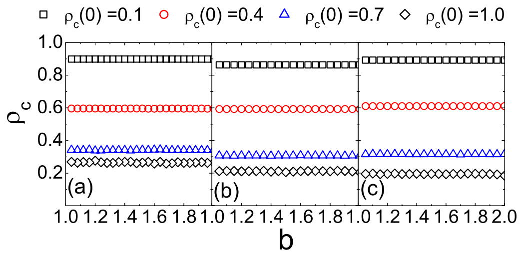


FIG. 2. (Color online). $\rho_c$ as a function of $r$ of SG on BA and NW networks. (a) BA networks with $k_{\min}=3$, (b) BA networks with $k_{\min}=6$, and (c) NW networks with $C_{NW}=2$, $P_{NW}=0.2$. For BA, $k_{\min}$ is the number of edges of the new nodes being added to the existent network at each time step. For NW, $C_{NW}$ is the coordination number of the initial ring network and $P_{NW}$ is the probability of each node to receive a new edge. $\rho_c(0)$ is the initial cooperator density. C and D are randomly distributed initially. Each data point is obtained by averaging over 100 simulations for each of ten different network realizations. The network size is 5000.

tive to the parameter $b$, but has a strong correlation with $\rho_c(0)$, as shown in Fig. 2, which indicates that there exist multistable states in the system. Cooperative behaviors on SG are displayed in Fig. 3. Compared to PDG, $\rho_c$ in SG exhibits not only multistable states but also discontinuous phase transition with varying $r$. The critical phase transition points are marked by $r_c$ in Fig. 3. Some steps are divided by the phase transition points and the values of $r_c$ are independent of $\rho_c(0)$. Within each step, $\rho_c$ is independent of $r$, while the value of each step depends on $\rho_c(0)$: the higher the $\rho_c(0)$, the higher the $\rho_c$. Analogous to the analysis of star networks, for inhomogeneous networks, high-degree nodes play a significant role in the organization of cooperation. We thus provide a prediction for $r_c$ by considering hub effects from a simple all-D initial state. In the initial state, all individuals' payoffs are zero, so they have the same death probability. For

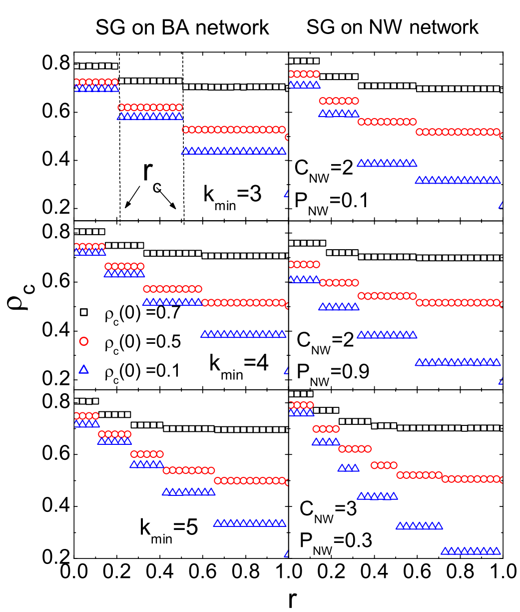


FIG. 3. (Color online). $\rho_c$ as a function of $r$ of SG on NW and BA networks. The phase transition points are marked by $r_c$. Other parameters are the same as those in Fig. 2.

BA networks, small-degree nodes are the majority, so small-degree individuals have high probabilities to change their strategies. Moreover, most neighbors of hubs are small-degree nodes. Hence after a small-degree individual turns to cooperator, hub nodes gain payoffs from it and their strategies will keep unchanged. As a result, hub individuals are more inactive than small-degree individuals, and the strategy changes of small-degree individuals mainly contribute to the change of $\rho_c$. Assume that the number of C neighbors of D individuals is $m_c$ and the minimum degree is $k_{\min}$. When $r$ approaches 1, the local pattern that all neighbors of a C individual with degree $k_{\min}$ are D individuals is steady. The payoff value of the C individual $(1-r)k_{\min}$ is the second least and if the individual with C switches to D, its payoff is zero, the least payoff value. Thus it will return to C. When some individuals with $k_{\min}$ adopt C, high-degree individuals of D will keep unchanged, since they have one or more small degree C neighbors and therefore gain payoffs. As $r$ decreases from 1, C individuals of degree $k_{\min}$ with only D neighbors according to the payoff matrix will gain more payoffs, which may be larger than the payoffs of D individuals with only one or more C neighbors $(1+r)m_c$. Thus the D individuals will change to C and the cooperator density will turn to another level, which is expressed as $k_{\min}(1-r)>m_c(1+r)$. This gives

$$r<\frac{k_{\min}-m_c}{k_{\min}+m_c}. \tag{2}$$

As one can see, when $k_{\min}-m_c<0$, i.e., $m_c>k_{\min}$, the high-degree D individuals always survive. At the critical values $r_c=(k_{\min}-m_c)/(k_{\min}+m_c)$, a discontinuous phase transition occurs. $m_c$ can be $1,2,\ldots,k_{\min}-1$, so that the number of steps is equal to $k_{\min}$. These results are confirmed by simulations, as shown in Fig. 3. For the NW networks, $k_{\min}$ is determined by the coordination number $C_{NW}$ of the initial regular ring since the NW network is constructed by randomly adding new edges to a regular ring. Thus $k_{\min}=2C_{NW}$ and the number of steps is $2C_{NW}$, as displayed in the right column of Fig. 3. We also notice that although the number of steps is independent of the small-world parameter $P_{NW}$, the parameter can influence the value of each step.

To better understand the effects of degree heterogeneity on the game dynamics, we investigate the average strategy degrees $\langle k_s\rangle$ of both C and D individuals for different values of $r$. As shown in Fig. 4, $\langle k_s\rangle$ has similar step structures and the sharp transition points are exactly the same as $\rho_c$. As $r$ decreases from 1, the average defector degree $\langle k_D\rangle$ of D individuals increases, which shows the same trend as $\rho_c$. On the contrary, the average cooperator degree $\langle k_c\rangle$ of C individuals displays a decreasing trend with the decrease of $r$. These findings are consistent with our analysis. For very large $r$, D individuals with small degrees are easier to switch to C. As $r$ reduces to another step, more small-degree D individuals will switch to C, which enlarges $\langle k_D\rangle$ and simultaneously reduces $\langle k_c\rangle$. When $r$ approaches 0, most individuals become C together with a few high-degree individuals adopting D. From our analysis, high-degree individuals tend to preserve their initial strategies because they usually gain more payoffs from many neighbors; while small-degree in-

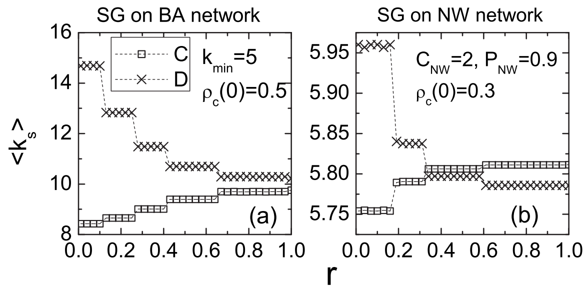


FIG. 4. Average strategy degrees $\langle k_s \rangle$ of C and D individuals for NW and BA networks. Symbol denotations and other parameters are the same with those in Fig. 2.

dividuals change strategies more easily and have to choose C to gain payoff $1-r$ from their D neighbors. Therefore it is the passive decision-making of small-degree nodes that leads to the domination of cooperation in the model, even when individuals are all defectors initially.

## V. HYSTERESIS LOOPS

Due to the trend of high-degree individuals preserving their initial strategies, the existence of multistable states in the system is indeed resulted from the difference of high-degree individuals' initial strategies. Interestingly, we found hysteresis loops of $\rho_c$ under some special conditions, which is partially due to the existence of multistable states. As shown in Fig. 5, there are two branches in each subset. The red-circle branch is the stable values of $\rho_c$ along the direction from $r=0$, while the black-square branch comes from very high values of $r$. These directions of varying $r$ are marked by arrows in Fig. 5. For each branch, after the system reaches a steady state under an $r$ value, the value of $r$ will be changed

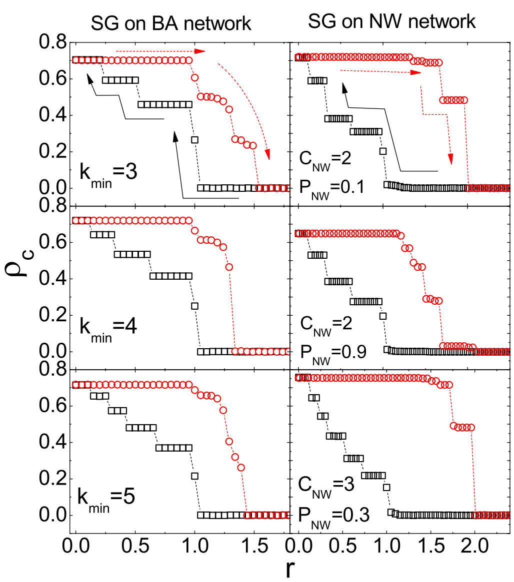


FIG. 5. (Color online). Hysteresis loops of $\rho_c$ for BA and NW networks. The initial cooperator density is $\rho_c(0)=0$. Arrows denote the directions of varying parameter $r$ for two branches. Symbol denotations and other parameters are the same as those in Fig. 2.

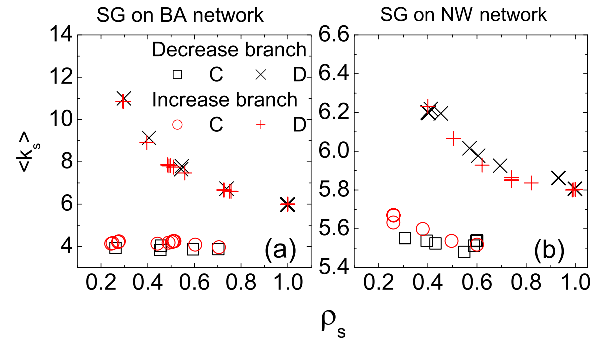


FIG. 6. (Color online). Average strategy degrees $\langle k_s \rangle$ depending on strategy density $\rho_s$ of C and D individuals for two different branches, respectively. (a) for BA network with $k_{\min}=3$ and (b) for NW network with $C_{NW}=2$ and $P_{NW}=0.9$. Symbol denotations and other parameters are the same as those in Fig. 5.

by a small amount $\delta r$ following the varying direction, and the strategies of individuals in the steady state are preserved as the initial state of the system for the new parameter value (for the increase branch the parameter value is $r+\delta r$ and for the decrease branch is $r-\delta r$). In the cases of small and large values of $r$, these two branches overlap, indicating that the system has the same dynamical properties along two different directions. In the region of medium values of $r$, a hysteresis loop can be traced out (arrows in Fig. 5), indicating that the system is bistable there. Moreover, the lower branch shows similar discontinuous phase transitions and step structures, where the phase transition points are the same as those shown in Fig. 3. The hysteresis loops and bistable states have been reported in magnetic systems and traffic systems [20] before, but as far as we know, they have not been found in network games prior to the present work.

In order to understand the existence of bistable states and the occurrence of hysteresis loops, we explore the average strategy degrees $\langle k_s \rangle$ of cooperators and defectors depending on the strategy density $\rho_s$ of cooperators and defectors for two branches, respectively. As shown in Fig. 6, for each branch, $\langle k_c \rangle$ versus $\rho_c$ and $\langle k_D \rangle$ versus $\rho_D$ display different behaviors, as reflected by two curves in Fig. 6. However, $\langle k_s \rangle$ versus $\rho_s$ of both branches shows a similar property for C and D, respectively, since the data points for the two branches overlap in the same curves. This result indicates that although varying the parameter along different directions results in different cooperative behaviors, the internal correlation, i.e., the behavior of $\langle k_s \rangle$ versus $\rho_s$, remains unchanged for two distinct branches. The major factor that plays the key role for the emergence of bistable states and hysteresis loops is the "memory effect." As shown in Fig. 3, the steady values of $\rho_c$ is determined by the initial cooperator density $\rho_c(0)$, and the higher the $\rho_c(0)$, the higher the steady $\rho_c$. In Fig. 5, for a value of $r$ on the increase branch, after the steady value $\rho_c$ is achieved, this steady cooperator density is then used as the initial state $\rho_c(0)$ for achieving a steady cooperative state under the condition $r+\delta r$. In other words, the system's cooperative behavior for a value of $r$ has a dependence on the former cooperator density along the direction of varying the parameter, i.e., the "memory effect." Since the increase and decrease branches are obtained from high and low cooperator densities initially, there exists a bistable range and in the range, $\rho_c$ of the increase branch is higher than that of the decrease branch due to the positive correlation between $\rho_c(0)$ and steady $\rho_c$.

## VI. CONCLUSION

In summary, we have studied the Prisoner's Dilemma game and snowdrift game with the global strategy updating mechanism on regular, small-world, and scale-free networks. We found that on these networks there exist multistable states for both the Prisoner's Dilemma game and snowdrift game, closely related with the initial cooperator density. In the snowdrift game, discontinuous phase transition occurs at some critical values of $r$, and step structures are divided by these phase transition points, which have been explained with respect to the significance of high-degree individuals in the system. The death of defectors with small degrees under the influence of high-degree individuals leads to the persistence and domination of cooperation. Finally, we have shown hysteresis loops of the cooperator density, obtained by varying parameter $r$ along two directions. Besides, we found similar internal correlation between strategy degrees and strategy densities of both branches in the hysteresis loops. The observed hysteresis property indicates that the dynamics of our game model in a certain region rely considerably on the historical cooperation behaviors.

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[1] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).
[2] S. N. Dorogovtsev and J. F. F. Mendes, Adv. Phys. **51**, 1079 (2002).
[3] M. E. J. Newman, SIAM Rev. **45**, 167 (2003).
[4] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, Phys. Rep. **424**, 175 (2006).
[5] J. Hofbauer and K. Sigmund, *Evolutionary Games and Population Dynamics* (Cambridge University Press, Cambridge, England, 1998).
[6] *Genetic and Cultural Evolution of Cooperation*, edited by P. Hammerstein (MIT, Cambridge, MA, 2003).
[7] G. Szabó and G. Fath, Phys. Rep. **446**, 97 (2007).
[8] M. Nowak and R. M. May, Nature (London) **359**, 826 (1992); M. Nowak and R. M. May, Int. J. Bifurcation Chaos Appl. Sci. Eng. **3**, 35 (1993); M. Doebeli and N. Knowlton, Proc. Natl. Acad. Sci. U.S.A. **95**, 8676 (1998); T. Killingback, M. Doebeli, and N. Knowlton, Proc. R. Soc. London, Ser. B **266**, 1723 (1999); T. Killingback and M. Doebeli, J. Theor. Biol. **191**(3), 335 (1998); C. Hauert and M. Doebeli, Nature (London) **428**, 643 (2004); M. Sysi-Aho, J. Saramäki, J. Kertész, and K. Kaski, Eur. Phys. J. B **44**, 129 (2005); L.-X. Zhong, D.-F. Zheng, B. Zheng, C. Xu, and P. M. Hui, Europhys. Lett. **76**, 724 (2006); Y.-S. Chen, H. Lin, and C.-X. Wu, Physica A **385**, 379 (2007); A. Szolnoki and G. Szabó, Europhys. Lett. **77**, 30004 (2007).
[9] G. Szabó and C. Hauert, Phys. Rev. Lett. **89**, 118101 (2002); F. C. Santos and J. M. Pacheco, *ibid.* **95**, 098104 (2005); G. Abramson and M. Kuperman, Phys. Rev. E **63**, 030901(R) (2001); B. J. Kim, A. Trusina, P. Holme, P. Minnhagen, J. S. Chung, and M. Y. Choi, *ibid.* **66**, 021907 (2002); H. Ebel and S. Bornholdt, *ibid.* **66**, 056118 (2002); Z.-X. Wu, X.-J. Xu, Y. Chen, and Y.-H. Wang, *ibid.* **71**, 037103 (2005); F. C. Santos, J. F. Rodrigues, and J. M. Pacheco, *ibid.* **72**, 056128 (2005); W.-X. Wang, J. Ren, G. Chen, and B.-H. Wang, *ibid.* **74**, 056113 (2006); C.-L. Tang, W.-X. Wang, X. Wu, and B.-H. Wang, Eur. Phys. J. B **53**, 411 (2006); J. Ren, W.-X. Wang, and F. Qi, Phys. Rev. E **75**, 045101(R) (2007); Z. Rong, X. Li, and X. Wang, *ibid.* **76**, 027101 (2007); J.-Y. Guan, Z.-X. Wu, and Y.-H. Wang, *ibid.* **76**, 056101 (2007); F. Fu, L.-H. Liu, and L. Wang, Eur. Phys. J. B **56**, 367 (2007); A. Cassar, Games Econ. Behav. **58**, 209 (2007); X. Chen and L. Wang, Phys. Rev. E **77**, 017103 (2008).
[10] M. A. Nowak, A. Sasaki, C. Taylor, and D. Fudenberg, Nature (London) **428**, 646 (2004).
[11] P. D. Taylor, T. Day, and G. Wild, Nature (London) **447**, 469 (2007).
[12] E. Lieberman, C. Hauert, and M. A. Nowak, Nature (London) **433**, 312 (2005).
[13] H. Ohtsuki, C. Hauert, E. Lieberman, and M. A. Nowak, Nature (London) **441**, 502 (2006).
[14] H. Ohtsuki, M. A. Nowak, and J. M. Pacheco, Phys. Rev. Lett. **98**, 108106 (2007); Z.-X. Wu and Y.-H. Wang, Phys. Rev. E **75**, 041114 (2007).
[15] J. M. Pacheco, A. Traulsen, and M. A. Nowak, Phys. Rev. Lett. **97**, 258103 (2006); M. O. Jackson, Games Econ. Behav. **51**, 128 (2005).
[16] S. Goyal and S. Joshi, Int. J. Game Theory **34**, 319 (2006); M. Perc and A. Szolnoki, Phys. Rev. E **77**, 011904 (2008).
[17] M. E. J. Newman and D. J. Watts, Phys. Rev. E **60**, 7332 (1999).
[18] D. J. Watts and S. H. Strogaz, Nature (London) **393**, 440 (1998).
[19] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).
[20] X. Li, Q. Wu, and R. Jiang, Phys. Rev. E **64**, 066128 (2001); M.-B. Hu, W.-X. Wang, R. Jiang, Q.-S. Wu, and Y.-H. Wu, *ibid.* **75**, 036102 (2007).